MATHEMATICS

A GENERALIZATION OF A THEOREM OF MARJANOVIC AND KIANG CONCERNING NONEXPANSIVE MAPS

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§ 1. INTRODUCTION

Our motivation for this paper comes from the work of Kiang [3], who, generalizing a result of Marjanovič [4] proved, under certain conditions, the equivalence of even continuity of a family of self-maps on a metric space with its nonexpansiveness. We generalize Kiang's theorem to uniform spaces, nonexpansiveness being in the sense of Brown and Comfort [1]. Our proof is much simpler than Kiang's, and moreover, clarifies the raison d'être for her hypothesis. Her result follows from ours via the well-known result, relating equicontinuity to even continuity, found in Kelley [2].

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§ 2. EQUICONTINUITY AND NONEXPANSIVENESS

We recall a few definitions and fix our terminology. Let X be any nonempty set and $\mathscr{B}$ a symmetric uniformity base on X.

2.1. DEFINITION. A family $\mathscr{F}$ of self-maps on X is said to be *$\mathscr{B}$-nonexpansive* iff for each $B \in \mathscr{B}$, $(x, y) \in B$ implies that for every $f \in \mathscr{F}$, $(f(x), f(y)) \in B$ (Cf. Brown and Comfort [1]).

2.2. DEFINITION. Let Z be a topological space. A family $\mathscr{F}$ of maps from Z to X is said to be *equicontinuous* iff for each $x \in Z$ and each $B \in \mathscr{B}$, there exists a neighbourhood (nhbd.) U of x such that for each $f \in \mathscr{F}$, $f(U) \subset B[f(x)]$.

For information on equicontinuity, uniform equicontinuity and even continuity, see Kelley [2].

If a family $\mathscr{F}$ of self-maps on X is $\mathscr{B}$-nonexpansive, then it is obviously (uniformly) equicontinuous. In the other direction, we have the following result:

2.3. LEMMA. Let $(X, \mathscr{U})$ be a uniform space and $\mathscr{F}$ be a semigroup of self-maps on X. If $\mathscr{F}$ is equicontinuous, then there exists a uniformity base $\mathscr{B}$, inducing the same topology as $\mathscr{U}$, such that $\mathscr{F}$ is $\mathscr{B}$-nonexpansive.

Further, if $\mathscr{F}$ is uniformly equicontinuous, then $\mathscr{B}$ may be chosen to be a base for $\mathscr{U}$ itself.

PROOF. Suppose $\mathscr{F}: X \to (X, \mathscr{U})$ is equicontinuous. For each $U \in \mathscr{U}$, set

$$U^* = \bigcap_{f \in \mathscr{F}} \{(x, y): (f(x), f(y)) \in U\} \cap U$$

and

$$\mathscr{B} = \{U^*: U \in \mathscr{U}\}.$$

It is easily verified that $\mathscr{B}$ is a uniformity base, and that the topology induced by $\mathscr{B}$ on X is finer than the one induced by $\mathscr{U}$. To show that the two topologies are equal, we verify that the $\mathscr{B}$-topology is also coarser than the $\mathscr{U}$-topology. Let $x \in X$ and $U^* \in \mathscr{B}$ for some $U \in \mathscr{U}$. Since $\mathscr{F}$ is equicontinuous, there exists a $V \in \mathscr{U}$ such that for every $f \in \mathscr{F}$, $f(V[x]) \subset U[f(x)]$. But then, for each $(x, y) \in U \cap V$ $(f(x), f(y)) \in U$, i.e. $(x, y) \in U^*$. Thus $U \cap V[x] \subset U^*[x]$.

To see that $\mathscr{F}$ is $\mathscr{B}$-nonexpansive, let $(x, y) \in U^* \in \mathscr{B}$ for some $U \in \mathscr{U}$, and let $f \in \mathscr{F}$. Then $(f(x), f(y)) \in U$, and for every $g \in \mathscr{F}$, $g \circ f \in \mathscr{F}$, and so $(g \circ f(x), g \circ f(y)) \in U$, whence $(f(x), f(y)) \in U^*$. The second assertion in the statement of the lemma follows from the construction of $\mathscr{B}$.

Thus we have shown:

2.4. THEOREM. Let X be a Tychonoff space, and $\mathscr{F}$ be a semigroup of self-maps on X. Then $\mathscr{F}$ is equicontinuous with respect to a compatible uniformity $\mathscr{U}$ on X if and only if $\mathscr{F}$ is $\mathscr{B}$-nonexpansive for some topologically equivalent uniformity base $\mathscr{B}$ on X.

Let $(X, \mathscr{U})$ be a uniform subspace of a uniform space $(Y, \mathscr{V})$, and let $\mathscr{F}$ be a family of self-maps on X. Then $\mathscr{F}: X \to (X, \mathscr{U})$ is equicontinuous iff $\mathscr{F}: X \to (Y, \mathscr{V})$ is equicontinuous. Moreover, if for each $x \in X$, $\mathscr{F}(x) = \{f(x): f \in \mathscr{F}\}$ has a compact closure in Y, then $\mathscr{F}: X \to (Y, \mathscr{V})$ is equicontinuous iff $\mathscr{F}$ is evenly continuous (Kelley [2], Page 237).

Thus we obtain from Theorem 2.4 the following generalization of the main result of Kiang [3]:

2.5. THEOREM. Let Y be a Tychonoff space, $X \subset Y$, and let $\mathscr{F}$ be a semigroup of self-maps on X such that for each $x \in X$, $\mathscr{F}(x)$ has a compact closure in Y (which is true if Y is compact). Then $\mathscr{F}: X \to Y$ is evenly continuous if and only if $\mathscr{F}$ is $\mathscr{B}$-nonexpansive for some topologically equivalent uniformity base $\mathscr{B}$ on X.

2.6. COROLLARY. If, in theorems 2.4 or 2.5, $\mathscr{F}$ is a group, then we may replace $\mathscr{B}$-nonexpansive by $\mathscr{B}$-isobasic (i.e. for each $B \in \mathscr{B}$, $(x, y) \in B$ iff $(f(x), f(y)) \in B$ for each $f \in \mathscr{F}$).

The above corollary includes the following result of Marjanovič [4]:

2.7. Corollary. Let f be a homeomorphism of a separable, locally compact, metrizable topological space M onto itself. Then f is a topological isometry if and only if the family $\{f^i: i \text{ an integer}\}$ of mappings of M into its one-point compactification M^* is evenly continuous.

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